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# Massive ghost theories with a line of defects

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**Abstract**

We study free massive fermionic ghosts, in the presence of an extended line of impurities, relying on the Lagrangian formalism. We propose two distinct defect interactions, respectively, of relevant and marginal nature. The corresponding scattering theories reveal the occurrence of resonances and instabilities in the former case and the presence of poles with imaginary residues in the latter. Correlation functions of the thermal and disorder operators are computed exactly, exploiting the bulk form factors and the matrix elements relative to the defect operator. In the marginal situation, the one-point function of the disorder operator displays a critical exponent continuously varying with the interaction strength.

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## 1. Introduction

After the seminal work by Ghoshal and Zamolodchikov [1] on integrable field theories in the presence of a boundary, a great deal of attention has been devoted to studying finite-size effects, due especially to their numerous applications to real physical problems. Quantum field theories with extended line of defects[1] generalize these boundary models, introducing new and original features [11–14].

The presence of impurities can be mimicked by the action of a ‘defect’ operator, placed along an infinite line in the Euclidean space. In the continuum limit and away from criticality, massive excitations can either participate in bulk scattering processes or interact with the defect. In general, due to the breaking of translational invariance, only reflection and transmission are allowed. Such information can be encoded into a scattering theory enriched by adding to the bulk S-matrix the amplitudes relative to these two new processes. The integrability of the model, originally studied in [11], is guaranteed by imposing the factorization condition which translates into a set of cubic relations called the reflection–transmission equations.

[1] Actually, the critical behaviour of statistical systems with lines of defects has been widely studied in the past few years [2–10].

In particular, it has been shown that, for diagonal bulk scattering, non-trivial solutions for both the reflection and transmission amplitudes can be found only in non-interacting bulk systems. In this light, free field theories play a prominent role.

Recently, wide interest has grown around free ghosts in two dimensions, due to their relevance to the study of disordered systems, polymer physics, quantum Hall states [15–19] and above all as an example of the simplest non-unitary/logarithmic conformal field theories [20, 21]. An exhaustive analysis of the fermionic and bosonic ghosts' conformal field theories, possessing respectively conformal charges $c = -2$ and $c = -1$, can be found in [22–24].

The main purpose of this work is to generalize a previously studied model of free massive fermionic ghosts [25], in order to include the effects of inhomogeneities. In particular, knowledge of the scattering amplitudes (and the spectrum of bulk excitations), along with general analyticity properties and relativistic invariance, allows us to reconstruct thoroughly the off-shell dynamics, by computing exactly correlation functions.

The first step towards the realization of this programme involves the derivation of the transmission and reflection amplitudes. One way to compute them consists in solving a bootstrap system of equations (unitarity, crossing and factorization). However, in this peculiar case, the absence of stringent constraints leaves a broad arbitrariness in the choice of the solutions. Fortunately, an alternative description is possible, in terms of the Lagrangian formalism

$$\mathcal{A} = \mathcal{A}_B + g \int \mathrm{d}^2 x \, \delta(x) \mathcal{L}_D(\varphi_i, \partial_y \varphi_i) \tag{1}$$

where the bulk Euclidean action $\mathcal{A}_B$ and the Lagrangian density $\mathcal{L}_D$, encoding all the information relative to the scattering processes on the defect line, both depend on the local fields of the theory. According to the strength of the coupling constant $g$, the line of inhomogeneities can interpolate between bulk and surface statistical behaviour. If the defect interaction is relevant (irrelevant), bulk (surface) behaviour is expected in the short-distance limit, while the marginal case shares both regimes. In the following, relevant and marginal interactions are proposed and exact expressions for the correlators of the most significant operators in the theory are derived, by using the bulk form factors and the matrix elements corresponding to the defect operator. In the former case, resonance phenomena occur in the spectrum of excitations, while the latter perturbation is responsible for non-universal power laws in the correlation functions of operators, non-local in the ghost fields.

## 2. Bootstrap approach

The model we are going to study is that of free massive fermionic ghosts [25] in the presence of an infinite line of impurities placed at $x = 0$, which, after a rotation in the Minkowski plane, will be identified with the time axis.

The bulk spectrum of the theory is composed of a doublet of free particles $A$ and $\bar{A}$ with mass $m$, bearing respectively U(1) charges $\pm 1$. Their scattering is ruled, in the bulk, by the S-matrix $S = -1$. Due to the energy conservation, when a particle hits the defect it can be either reflected or transmitted. All the processes involved in the theory can be recast as a set of algebraic equations [11], relying on the algebra of the Faddeev–Zamolodchikov operators. After the usual parametrization of the particle's energy–momentum in terms of the rapidity variable $(e, p) = (m \cosh \theta, m \sinh \theta)$, we associate with excitations of type '$a$' the formal operator $A_a(\theta)$ and with the defect line an operator $\mathcal{D}$, playing the role of a zero rapidity particle, during the whole time evolution of the system. The commutation relations, associated with the defect algebra, read

$$A_a(\theta)\mathcal{D} = R_a^b(\theta)A_b(-\theta)\mathcal{D} + T_a^b(\theta)\mathcal{D}A_b(\theta)$$
$$\mathcal{D}A_a(\theta) = R_a^b(-\theta)\mathcal{D}A_b(-\theta) + T_a^b(-\theta)A_b(\theta)\mathcal{D} \tag{2}$$

where, in the first equation, $R_a^b(\theta)$ and $T_a^b(\theta)$ denote, respectively, the reflection and transmission amplitudes of an asymptotic particle '$a$' entering the defect with rapidity $\theta$, from the left. The second equation, describing the scattering of a particle hitting the defect from the right, is obtained from the first one, after an analytic continuation $\theta \to -\theta$ in the rapidity variable. Consistency of (2) implies the unitarity conditions

$$R_a^b(\theta)R_b^c(-\theta) + T_a^b(\theta)T_b^c(-\theta) = \delta_a^c \qquad R_a^b(\theta)T_b^c(-\theta) + T_a^b(\theta)R_b^c(-\theta) = 0. \tag{3}$$

Crossing relations read

$$\mathcal{C}^{aa''}R_{a''}^b\left(\mathrm{i}\frac{\pi}{2} - \theta\right) = S_{a'b'}^{ab}(2\theta)\mathcal{C}^{b'b''}R_{b''}^{a'}\left(\mathrm{i}\frac{\pi}{2} + \theta\right) \qquad T_a^b(\theta) = \mathcal{C}^{bb'}T_{b'}^{a'}(\mathrm{i}\pi - \theta)\mathcal{C}_{a'a} \tag{4}$$

with an antisymmetric charge conjugation matrix, such that $\mathcal{C}^2 = -1$. As regards factorization conditions, the main result of [11] guarantees that, for free theories diagonal in the bulk, the reflection–transmission equations, descending from integrability, are automatically satisfied.

At this point, solving the bootstrap system of equations (2)–(4), we are able in principle to determine the scattering amplitudes $R_a^b$ and $T_a^b$. However, a proliferation of solutions occurs, due to the lack of constraints strong enough to fix the reflection and transmission matrices in a closed form. A simplified version of this model (i.e. a purely reflecting theory which coincides with a boundary problem [1]) helps in visualizing the situation. We introduce the following parametrization of the reflection matrix components:

$$R_A^A(\theta) = f(\theta)R(\theta) \qquad R_A^{\bar{A}}(\theta) = g(\theta)R(\theta)$$
$$R_{\bar{A}}^{\bar{A}}(\theta) = f'(\theta)R(\theta) \qquad R_{\bar{A}}^A(\theta) = g'(\theta)R(\theta). \tag{5}$$

Consistency of the bootstrap system gives rise to the conditions

$$R(\theta)R(-\theta) = [f(\theta)f(-\theta) + g(\theta)g'(-\theta)]^{-1}$$
$$R(\theta)R(-\theta) = [f'(\theta)f'(-\theta) + g'(\theta)g(-\theta)]^{-1} \tag{6}$$

$$f(\theta)g(-\theta) + g(\theta)f'(-\theta) = 0 \qquad f'(\theta)g'(-\theta) + g'(\theta)f(-\theta) = 0 \tag{7}$$

$$-\frac{g'\left(\frac{\mathrm{i}\pi}{2} + \theta\right)}{g'\left(\frac{\mathrm{i}\pi}{2} - \theta\right)} = \frac{f'\left(\frac{\mathrm{i}\pi}{2} + \theta\right)}{f\left(\frac{\mathrm{i}\pi}{2} - \theta\right)} = \frac{f\left(\frac{\mathrm{i}\pi}{2} + \theta\right)}{f'\left(\frac{\mathrm{i}\pi}{2} - \theta\right)} = -\frac{g\left(\frac{\mathrm{i}\pi}{2} + \theta\right)}{g\left(\frac{\mathrm{i}\pi}{2} - \theta\right)} \tag{8}$$

which allow a richness of solutions. A comparison with the well-established theory of free massive Dirac fermions [1, 26], showing strong analogies with our ghost system, is in order. Such a model, obtained as a particular limit of the sine-Gordon one, admits non-trivial boundary Yang–Baxter equations, which provide a solution for the reflection amplitude in terms of two parameters. In our case, starting directly from a free theory, it is impossible to exploit factorization constraints, in order to fix the form of the $R$-matrix.

## 3. Lagrangian description

To overcome the ambiguities, intrinsically concerned with the bootstrap scenario, the Lagrangian approach proves to be an alternative route.

The Euclidean action, describing the bulk dynamics, is that of free massless symplectic fermions [23], supplemented by a mass term

$$\mathcal{A}_B = \frac{1}{2}\int \mathrm{d}^2x\, J_{\alpha\beta}(\partial_\mu\Phi^\alpha\partial^\mu\Phi^\beta + m^2\Phi^\alpha\Phi^\beta). \tag{9}$$

$\Phi^\alpha$, which are zero-dimensional anti-commuting fields ($\Phi$ and $\bar{\Phi}$), belong to the same doublet, characterized by mass $m$, while $J_{\alpha\beta}$ is an antisymmetric tensor. A detailed analysis of the bulk system, including mode expansions of the basic fields, commutation relations and charge conjugation properties can be found in appendix A.

Inhomogeneities affect the bulk physics introducing a Lagrangian density along the impurity line, according to (1). A relevant and a marginal interaction will be the object of our study in order to derive explicit expressions for the reflection and transmission amplitudes.

### 3.1. Relevant perturbation

Consider the system described by

$$\mathcal{A} = \mathcal{A}_B + \frac{g}{2}\int \mathrm{d}^2x\,\delta(x)J_{\alpha\beta}\Phi^\alpha\Phi^\beta \tag{10}$$

where the dimension of the coupling constant $g$ is [mass]. The equations of motion read

$$(\partial_\mu\partial^\mu - m^2)\Phi = g\delta(x)\Phi \qquad (\partial_\mu\partial^\mu - m^2)\bar{\Phi} = g\delta(x)\bar{\Phi}. \tag{11}$$

It is useful to split the fields into components belonging to the two intervals $x < 0$ and $x > 0$ (after rotation to the Minkowski space)

$$\begin{aligned}\Phi(x,t) &= \theta(x)\Phi_+(x,t) + \theta(-x)\Phi_-(x,t)\\ \bar{\Phi}(x,t) &= \theta(x)\bar{\Phi}_+(x,t) + \theta(-x)\bar{\Phi}_-(x,t)\end{aligned} \tag{12}$$

in order to derive the boundary conditions at $x = 0$, given by

$$\Phi_+(0,t) - \Phi_-(0,t) = 0 \qquad \partial_x(\Phi_+(0,t) - \Phi_-(0,t)) = \frac{g}{4}(\Phi_+(0,t) + \Phi_-(0,t)) \tag{13}$$

$$\bar{\Phi}_+(0,t) - \bar{\Phi}_-(0,t) = 0 \qquad \partial_x(\bar{\Phi}_+(0,t) - \bar{\Phi}_-(0,t)) = \frac{g}{4}(\bar{\Phi}_+(0,t) + \bar{\Phi}_-(0,t)). \tag{14}$$

The mode expansions (65), in terms of the operators $A$ and $\bar{A}$ which interpolate the bulk excitations, allow us to extract explicitly from (13), (14) the reflection and transmission amplitudes

$$\begin{pmatrix} A_-(\beta)\\ \bar{A}_-(\beta)\\ A_+(-\beta)\\ \bar{A}_+(-\beta)\end{pmatrix} = \begin{pmatrix} R(\beta,\kappa) & T(\beta,\kappa)\\ T(\beta,\kappa) & R(\beta,\kappa)\end{pmatrix}\begin{pmatrix} A_-(-\beta)\\ \bar{A}_-(-\beta)\\ A_+(\beta)\\ \bar{A}_+(\beta)\end{pmatrix} \tag{15}$$

with

$$R(\beta,\kappa) = \frac{1}{\sinh\beta + \mathrm{i}\kappa}\begin{pmatrix} -\mathrm{i}\kappa & 0\\ 0 & -\mathrm{i}\kappa\end{pmatrix} \qquad T(\beta,\kappa) = \frac{1}{\sinh\beta + \mathrm{i}\kappa}\begin{pmatrix}\sinh\beta & 0\\ 0 & \sinh\beta\end{pmatrix} \tag{16}$$

and $\kappa = g/4m$. $R$ and $T$, thus obtained, satisfy crossing and unitarity conditions.

A strong analogy with the free bosonic theory, extensively treated in [11], emerges. Apart from a doubling of the matrix elements, the scattering amplitudes coincide. The main features are the occurrence of resonances (i.e. unstable bound states possessing a real part in the unphysical sheet, which do not appear as asymptotic particles of the theory) for $\kappa > 1$ and phenomena of instabilities for $\kappa < -1$, characterized by poles with imaginary part fixed at the value $\mathrm{i}\pi/2$, acquiring an increasing real part as $\kappa$ is further depleted.

In the limit $g \to \infty$ ($\kappa \to \infty$), corresponding to the fixed boundary conditions $\Phi(0,t) = 0$ and $\bar{\Phi}(0,t) = 0$, the defect line acts as a purely reflecting surface. In contrast, in the high-energy limit $\beta \to \infty$, due to the relevant character of the perturbation, the theory renormalizes to a bulk regime, the impurity line becoming transparent.

### *3.2. Marginal perturbation*

The Euclidean action

$$\mathcal{A} = \mathcal{A}_B - \mathrm{i}g \int \mathrm{d}^2x\,\delta(x)(\Phi\partial_y\Phi + \bar{\Phi}\partial_y\bar{\Phi}) \tag{17}$$

where $g$ is a dimensionless coupling constant describes the effects of a marginal interaction on the defect line. The equations of motion

$$(\partial_\mu\partial^\mu - m^2)\bar{\Phi} - 2\mathrm{i}g\delta(x)\partial\Phi = 0 \tag{18}$$

$$(\partial_\mu\partial^\mu - m^2)\Phi + 2\mathrm{i}g\delta(x)\partial\bar{\Phi} = 0 \tag{19}$$

lead to the following boundary conditions in the Minkowski plane:

$$\bar{\Phi}_+(0,t) - \bar{\Phi}_-(0,t) = 0 \qquad \partial_x(\bar{\Phi}_+(0,t) - \bar{\Phi}_-(0,t)) = g\partial_t\Phi(0,t) \tag{20}$$

$$\Phi_+(0,t) - \Phi_-(0,t) = 0 \qquad \partial_x(\Phi_+(0,t) - \Phi_-(0,t)) = -g\partial_t\bar{\Phi}(0,t). \tag{21}$$

Exploiting again the mode expansions in terms of the operators $A$ and $\bar{A}$, the reflection and transmission matrices assume the form

$$\begin{aligned} R(\beta,\chi) &= \frac{\sin\chi\cosh\beta}{\cosh^2\beta - \cos^2\chi}\begin{pmatrix} -\sin\chi\cosh\beta & -\cos\chi\sinh\beta \\ \cos\chi\sinh\beta & -\sin\chi\cosh\beta \end{pmatrix} \\ T(\beta,\chi) &= \frac{\cos\chi\sinh\beta}{\cosh^2\beta - \cos^2\chi}\begin{pmatrix} \cos\chi\sinh\beta & -\sin\chi\cosh\beta \\ \sin\chi\cosh\beta & \cos\chi\sinh\beta \end{pmatrix} \end{aligned} \tag{22}$$

$$\sin^2\chi = \frac{g^2}{4+g^2}. \tag{23}$$

Some remarks are in order. Action (17) is invariant under charge conjugation, implemented by the transformations $\Phi \to \bar{\Phi}$ and $\bar{\Phi} \to -\Phi$. Therefore, the relations $R_A^A = R_{\bar{A}}^{\bar{A}}$ and $R_A^{\bar{A}} = -R_{\bar{A}}^A$, along with their analogous counterparts for the transmission matrix, hold. On the other hand, the U(1) symmetry, manifestly displayed by the bulk action, is broken by the defect interaction. As a consequence, scattering processes, which violate the conservation of U(1) charges on the impurity line, can occur, allowing for non-zero off-diagonal contributions. Exceptions to this behaviour concern the fixed ($g \to \infty$, $\cos\chi \to 0$) and the free ($g \to 0$, $\sin\chi \to 0$) boundary conditions, where a restoration of the symmetry takes place.

Let us turn our attention to the analytic structure of the reflection and transmission matrices. Since the theory is non-unitary, a mechanism, akin to the one occurring in the scaling Lee–Yang model [27], is expected to take place. In other words, residues, corresponding to poles in the scattering amplitudes, are not supposed to be, *a priori*, real and positive. This phenomenon is reminiscent of the non-Hermitian nature of the Hamiltonian associated with the system and does not contrast with the unitarity requirement (3), preserving the meaning of probability densities[2].

[2] Non-Hermiticity of the Hamiltonian implies, in particular, that its left eigenstates $\langle n_L|$ are not simply the adjoints of the right ones $|n_R\rangle$. Since, in addition, the Fock space states are also eigenstates of the charge-conjugation operator with eigenvalues $(\pm\mathrm{i})^N$, $N$ being the number of particles, the relation $\langle n_L| = \langle n_R|\mathcal{C}$ leads to the completeness condition $\sum_n |n_R\rangle\langle n_L| = \sum_n |n_R\rangle\langle n_R|(\pm\mathrm{i})^n$. On the other hand, equation (3), relying only on the assumption that in and out-kets, constructed in terms of the asymptotic particles $A$ and $\bar{A}$, form a basis in the Hilbert space, is insensitive to Hermiticity properties of the Hamiltonian.

Poles appear both in the reflection and in the transmission amplitudes at $\beta = \mathrm{i}\chi$ and $\beta = \mathrm{i}(\pi - \chi)$, with $\chi \in [0, \pi/2]$. In the case of diagonal matrix elements, the corresponding residues give

$$\begin{aligned} R_A^A \simeq R_{\bar{A}}^{\bar{A}} \simeq T_A^A \simeq T_{\bar{A}}^{\bar{A}} \simeq \frac{\mathrm{i}}{2} \frac{\sin\chi\cos\chi}{\beta - \mathrm{i}\chi} \\ R_A^A \simeq R_{\bar{A}}^{\bar{A}} \simeq T_A^A \simeq T_{\bar{A}}^{\bar{A}} \simeq \frac{\mathrm{i}}{2} \frac{-\sin\chi\cos\chi}{\beta - \mathrm{i}(\pi - \chi)}. \end{aligned} \tag{24}$$

Therefore, the pole at $\beta = \mathrm{i}\chi$ is associated with a boundary bound state in the direct channel, with positive binding energy $e_b = m\cos\chi$, while the other one lives in the crossed channel. Since $e_b < m$ for every value of the coupling constant, the boundary bound states are always stable and the theory is free of resonances and instabilities of other nature. As regards off-diagonal processes, the residues calculated at $\beta = \mathrm{i}\chi$ assume the form

$$R_A^{\bar{A}} \simeq T_A^{\bar{A}} \simeq \frac{\mathrm{i}}{2} \frac{\mathrm{i}\sin\chi\cos\chi}{\beta - \mathrm{i}\chi} \qquad R_{\bar{A}}^A \simeq T_{\bar{A}}^A \simeq \frac{\mathrm{i}}{2} \frac{-\mathrm{i}\sin\chi\cos\chi}{\beta - \mathrm{i}\chi} \tag{25}$$

while residues computed in the crossed channel display an overall minus sign. As mentioned before, the additional factor $\pm\mathrm{i}$, appearing in the numerator, is a consequence of the anomalous charge conjugation properties of the ghost fields.

Finally, a comment on the marginal nature of the interaction: performing the ultraviolet limit, except for peculiar values of the coupling constant, all the scattering matrices' components remain simultaneously finite.

## 4. Correlation functions

The problem at the heart of this paper concerns the computation of correlation functions of the local fields $\phi_i(x, t)$, present in the theory.

To realize this idea, in order to fully exploit the knowledge of the bulk physics, it is worth performing a rotation in the Minkowski plane ($x \to -\mathrm{i}t, t \to \mathrm{i}x$), moving the defect line at $t = 0$. In this new picture, the Hilbert space of states is the same as in the bulk and the effects of impurities are taken into account by an operator $\mathcal{D}$, placed at $t = 0$, which acts on the bulk states. Therefore, correlation functions assume the form [11]

$$\langle \Phi_1(x_1, t_1) \dots \Phi_n(x_n, t_n) \rangle = \frac{\langle 0 | T[\phi_1(x_1, t_1) \dots \mathcal{D} \dots \phi_n(x_n, t_n)] | 0 \rangle}{\langle 0 | \mathcal{D} | 0 \rangle} \tag{26}$$

$\Phi_i(x_i, t_i)$ and $\phi_i(x_i, t_i)$ being the fields in the Heisenberg representation, whose time evolutions are ruled, respectively, by the exact Hamiltonian of the problem (bulk and defect interactions) and the bulk Hamiltonian alone. As appears clearly, after inserting the completeness condition of the bulk states in the right-hand side of (26), the above equation can be computed only in terms of the form factors of the bulk fields and the matrix elements of the defect operator on the asymptotic states. Another consequence of the axis rotation in the Minkowski plane is the interchange of roles between energy and momentum. This affects the rapidity dependence of the scattering amplitudes according to $\theta \to (\mathrm{i}\pi/2 - \theta)$. In compact notation it reads

$$\hat{R}^{ab}(\theta) = \mathcal{C}^{aa'} R_{a'}^b \left(\mathrm{i}\frac{\pi}{2} - \theta\right) \qquad \hat{T}^{ab}(\theta) = \mathcal{C}^{aa'} T_{a'}^{b'} \left(\mathrm{i}\frac{\pi}{2} - \theta\right) \mathcal{C}_{b'b}. \tag{27}$$

Let us recall here that asymptotic states are composed of neutral pairs $A(\theta)\bar{A}(\beta)$, obtained by acting with the corresponding operators $A$ and $\bar{A}$ on the vacuum $|0\rangle$. Explicit expressions for the bulk form factors have been derived in [25], while the simplest matrix elements of the defect operator on the bulk states are

$$\begin{aligned}
\langle A(\theta)|\mathcal{D}|A(\theta')\rangle &= 2\pi \hat{T}^{AA}(\theta)\delta(\theta-\theta') \\
\langle \bar{A}(\beta)A(\theta)|\mathcal{D}|0\rangle &= 2\pi \hat{R}^{A\bar{A}}(\theta)\delta(\theta+\beta) \\
\langle 0|\mathcal{D}|A(\theta)\bar{A}(\beta)\rangle &= -2\pi \hat{R}^{A\bar{A}}(\theta-\mathrm{i}\pi)\delta(\beta+\theta-2\pi\mathrm{i}).
\end{aligned} \tag{28}$$

In the remaining part of this section, we are going to study correlators of the operator

$$\omega(x,t) = \frac{J_{\alpha\beta}}{2}\Phi^{\alpha}\Phi^{\beta}(x,t) \tag{29}$$

associated with the massive perturbation of the critical bulk theory, and the one-point function of the 'disorder' operator $\mu$.

### *4.1. ω operator*

The simplest correlation function involving $\omega$ is the one-point function, defined as

$$\omega_0(t,g) \equiv \langle \omega(x,t)\rangle = \sum_{n=0}^{\infty}\langle 0|\omega(x,t)|n\rangle\langle n|\mathcal{D}|0\rangle \tag{30}$$

the resolution of the identity explicitly reading

$$1 = \sum_{n=0}^{\infty}\frac{1}{(n!)^2(2\pi)^{2n}}\int_{-\infty}^{+\infty}\mathrm{d}\theta_1\ldots\mathrm{d}\beta_n|A(\theta_1),\ldots,\bar{A}(\beta_n)\rangle\langle\bar{A}(\beta_n),\ldots,A(\theta_1)|. \tag{31}$$

Since $\omega$ is the operator perturbing the critical theory in the bulk, it turns out to be proportional to the trace of the stress–energy tensor [28]. This implies, for free theories, the remarkable property that only two-particle states can be coupled to the vacuum

$$\langle 0|\omega(x,t)|A(\theta_1)\bar{A}(\beta_1)\rangle = 2\pi\,\mathrm{e}^{-mt(\cosh\beta_1+\cosh\theta_1)+\mathrm{i}mx(\sinh\beta_1+\sinh\theta_1)}. \tag{32}$$

Thus, exploiting (28), $\omega_0$ can be recast as

$$\omega_0(t,g) = 2\int_0^{\infty}\mathrm{d}\theta\,\hat{R}^{A\bar{A}}(\theta)\,\mathrm{e}^{-2mt\cosh\theta}. \tag{33}$$

Such a formula is amenable to discuss the different defect interactions.

For free boundary conditions, the reflection matrix is trivially zero and the one-point function vanishes. In the case of fixed boundary conditions, instead, $\hat{R}^{A\bar{A}}(\theta) = -1$ and the short-distance limit is easily derived,

$$\omega_0(t) = -2\int_0^{\infty}\mathrm{d}\theta\,\mathrm{e}^{-2mt\cosh\theta} = -2K_0(2mt) \to 2\ln(mt) \qquad mt\to 0. \tag{34}$$

Concerning the relevant perturbation, (33) assumes the form

$$\omega_0(t,\kappa) = -\kappa\int_{-\infty}^{+\infty}\mathrm{d}\theta\frac{\exp[-2mt\cosh\theta]}{\cosh\theta+\kappa}. \tag{35}$$

In the limit of fixed boundary conditions ($\kappa\to\infty$) the previous result (34) naturally follows while, in order to study the large- and short-distance regimes for arbitrary $\kappa$, it could be meaningful to manipulate a little bit the expression of $\omega_0$. The differential equation

$$\frac{\partial\omega_0(t,\kappa)}{\partial(2mt)} - \kappa\omega_0(2mt,\kappa) = 2\kappa K_0(2mt) \tag{36}$$

descending from (35), helps in deducing the large-distance limit. Substituting the trial expansion $\omega_0(t,\kappa) \sim \mathrm{e}^{-2mt}(2mt)^{-\gamma}\sum a_l(2mt)^{-l}$ into it, the asymptotic behaviour

$\omega_0 \sim \frac{-2\kappa}{1+\kappa} K_0(2mt)$ is recovered as $mt \to \infty$. On the other hand, the ultra-violet limit emerges more clearly if we look at the expression

$$\omega_0(t,\kappa) = -2\kappa\, \mathrm{e}^{(2mt)\kappa} \int_{2mt}^{\infty} \mathrm{d}\eta\, \mathrm{e}^{-\eta\kappa} K_0(\eta). \tag{37}$$

As far as $mt \to 0$, $\omega_0$ always assumes finite values. Summarizing, in the infra-red regime $\omega_0$ follows the asymptotic behaviour typical of the fixed boundary conditions, while for small distances it remains finite, approaching zero as the coupling constant vanishes.

An analogous analysis can be performed for the marginal interaction. The one-point function (33) specializes to

$$\omega_0(t,\chi) = -\sin^2\chi \int_{-\infty}^{+\infty} \mathrm{d}\theta \frac{\sinh^2\theta}{\cosh^2\theta - \sin^2\chi}\, \mathrm{e}^{-2mt\cosh\theta}. \tag{38}$$

The corresponding differential equation

$$\frac{\partial^2\omega_0(t,\chi)}{\partial(2mt)^2} - \sin^2\chi\,\omega_0(t,\chi) = -2\sin^2\chi \frac{K_1(2mt)}{2mt} \tag{39}$$

allows us to derive both the asymptotic limits. Exploiting a series expansion, as we did in the relevant case, the low-energy regime leads to two different types of behaviour

$$\begin{cases} \omega_0(t,\chi) \to \ \mathrm{e}^{-(2mt)}(2mt)\sqrt{\frac{\pi}{2}\frac{2\sin^2\chi}{\sin^2\chi-1}} & \sin^2\chi \neq 1 \\ \omega_0(t,\chi) \to -2K_0(2mt) & \sin^2\chi = 1. \end{cases} \tag{40}$$

As regards the ultra-violet limit, $\omega_0(t,\chi) \sim 2\sin^2\chi\,\ln(2mt)$.

We turn now the attention to the two-point functions involving the operator $\omega$. Two different situations can occur.

Consider the case in which the operators lie on opposite sides of the defect line, i.e. $t_1 < 0$ and $t_2 > 0$. The correlator is given by

$$G_1(\rho_1,\rho_2;g) = \sum_{i,j} \langle 0|\omega(\rho_2)|i\rangle\langle i|\mathcal{D}|j\rangle\langle j|\omega(\rho_1)|0\rangle \tag{41}$$

with the collective variable $\rho = (x,t)$. As before, the series contains only a finite number of terms. In order to perform the calculations, we need the expression of the 'defect' matrix element involving four particles

$$\begin{aligned} \langle\bar{A}(\beta_1)A(\theta_1)|\mathcal{D}|A(\theta_1')\bar{A}(\beta_1')\rangle &= (2\pi)^2[\hat{T}^{AA}(\theta_1)\hat{T}^{\bar{A}\bar{A}}(\beta_1)\delta(\theta_1-\theta_1')\delta(\beta_1-\beta_1') \\ &\quad - \hat{R}^{A\bar{A}}(\theta_1)\hat{R}^{A\bar{A}}(\theta_1'-\mathrm{i}\pi)\delta(\beta_1+\theta_1)\delta(\beta_1'+\theta_1'-2\pi\mathrm{i})]. \end{aligned} \tag{42}$$

Introducing a redefinition of variables in terms of $t \equiv t_2 - t_1$ and $x \equiv x_2 - x_1$, we finally obtain

$$G_1(\rho_1,\rho_2;\kappa) = -\left[\frac{\partial F(mx,mt;\kappa)}{\partial(mt)}\right]^2 + \omega_0(t_1,\kappa)\omega_0(t_2,\kappa) \tag{43}$$

$$F(x,t) = \int_{-\infty}^{+\infty} \mathrm{d}\theta \frac{\exp[-t\cosh\theta + \mathrm{i}x\sinh\theta]}{\cosh\theta + \kappa} \tag{44}$$

for the relevant perturbation and

$$G_1(\rho_1,\rho_2;\chi) = -\cos^4\chi \left[\frac{\partial^2 F(mx,mt;\chi)}{\partial(mt)^2}\right]^2 + \omega_0(t_1,\chi)\omega_0(t_2,\chi) \tag{45}$$

$$F(x,t) = \int_{-\infty}^{+\infty} \mathrm{d}\theta \frac{\exp[-t\cosh\theta + \mathrm{i}x\sinh\theta]}{\cosh^2\theta - \sin^2\chi} \tag{46}$$

for the marginal one. In the limit of an infinitely reflecting surface ($\kappa \to \infty$ and $\cos^2\chi \to 0$), only the vacuum expectation values of the two $\omega$ operators survive.

Another situation can happen, in which the two $\omega$ operators reside on the same half of the Minkowski plane. Let us consider, for convenience, $t_2 \geqslant t_1 > 0$ and define $t \equiv t_2 - t_1$, $\bar{t} \equiv t_2 + t_1$, $x \equiv x_2 - x_1$, $r \equiv \sqrt{x^2 + t^2}$. The general expression for the two-point function is

$$G_2(\rho_1, \rho_2; g) = \sum_{i,j} \langle 0|\omega(\rho_2)|i\rangle\langle i|\omega(\rho_1)|j\rangle\langle j|\mathcal{D}|0\rangle. \tag{47}$$

Following the lines traced in [11], after straightforward calculations, we end up with

$$G_2(\rho_1, \rho_2; \kappa) = -[2K_0(mr) + \kappa F(m\bar{t}, mx)]^2 + \omega_0(t_1, \kappa)\omega_0(t_2, \kappa) \tag{48}$$

in the relevant case and

$$G_2(\rho_1, \rho_2; \chi) = -\left[2K_0(mr) + \sin^2\chi \frac{\partial^2 F(m\bar{t}, mx)}{\partial(mx)^2}\right]^2 + \omega_0(t_1, \chi)\omega_0(t_2, \chi) \tag{49}$$

for the marginal perturbation. As appears clearly, the solutions found are invariant under translations along the $x$-axis, consistent with the picture adopted, which preserves momentum.

### 4.2. Disorder operator

Finally, we examine the one-point function of the operator $\mu$, which is only a specific example belonging to the widest class of the ‘disorder’ operators, non-local with respect to the ghost fields. The analysis concerning the leading behaviour of their correlators, which relies on a ‘cluster’ expansion, is the main purpose of appendix C, while a detailed discussion about them in bulk free theories can be found in [29–31] (ordinary complex fermions and bosons) and [25] (fermionic and bosonic ghost systems). The one-point correlator can be written as follows:

$$\mu_0(t, g) \equiv \langle \mu(x, t)\rangle = \sum_n \langle 0|\mu(x, t)|n\rangle\langle n|\mathcal{D}|0\rangle. \tag{50}$$

In this case, $\mu$ couples the vacuum to neutral states, composed of an even number of excitations. As a consequence, the sum does not truncate and, to explicitly evaluate (50), matrix elements involving an arbitrary (even) number of particles

$$\langle \bar{A}(\beta_n)\ldots\bar{A}(\beta_1), A(\theta_n)\ldots A(\theta_1)|\mathcal{D}|0\rangle = (-)^{\frac{n(n-1)}{2}} n!(2\pi)^n \prod_{k=0}^{n} \hat{R}^{A\bar{A}}(\beta_k)\delta(\beta_k + \theta_k) \tag{51}$$

are required. In addition, since the defect operator $\mathcal{D}$ is responsible for processes involving only absorption or emission of couples of particles with opposite rapidities, $\mu_0$ finally assumes the form

$$\mu_0(t, g) = \sum_{n=0}^{\infty} \frac{(-)^{n(n-1)/2}}{n!} \int \frac{d\beta_1}{2\pi}\ldots\frac{d\beta_n}{2\pi} \prod_{k=1}^{n} [\hat{R}^{A\bar{A}}(\beta_k)\, e^{-2mt\cosh\beta_k}]$$
$$\times f_n^{1/2}(-\beta_1, \ldots, -\beta_n, \beta_1, \ldots, \beta_n). \tag{52}$$

Exact expressions for the bulk form factors are given in [25]

$$f_n^{1/2}(\theta_1, \ldots, \theta_n, \beta_1, \ldots, \beta_n) = \langle 0|\mu_{1/2}(0)|A(\theta_1)\ldots A(\theta_n)\bar{A}(\beta_1)\ldots\bar{A}(\beta_n)\rangle$$
$$= (-)^{n(n+1)/2}|A_n| \tag{53}$$

where $|A_n|$ denotes the determinant of a matrix whose components read

$$A_{ij} = \frac{1}{\cosh\frac{\theta_i - \beta_j}{2}}. \tag{54}$$

In order to discuss the effects due to the different interactions localized along the defect line, (52) proves to be a good starting point.

Again, free boundary conditions lead to the trivial solution $\mu_0 = 0$. In the case of fixed boundary conditions, it is possible to recover the leading short-distance behaviour of the one-point function, in an exact way. The details of the calculation will be postponed to appendix B, while here only the main results will be given. Since the reflection matrix component $\hat{R}^{A\bar{A}}$ is trivially $-1$, exploiting the theory of Fredholm determinants [32], $\mu_0$ can be recast as

$$\mu_0(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\theta_1}{2\pi} \cdots \frac{\mathrm{d}\theta_n}{2\pi} \, \mathrm{e}^{-2mt \sum_1^n \cosh\theta_k} |A_n| = \det\left(1 + \frac{1}{\pi} V(t)\right) \tag{55}$$

where the kernel is given explicitly by

$$V(\theta_i, \theta_j, t) = \frac{e(\theta_i, t) e(\theta_j, t)}{2\cosh\frac{\theta_i+\theta_j}{2}} \qquad \mathrm{e}(\theta, t) = \mathrm{e}^{-mt\cosh\theta}. \tag{56}$$

Alternatively, $\mu_0$ can be expressed as

$$\mu_0(t) = \prod_{i=1}^{\infty} \left(1 + \frac{1}{\pi}\lambda^{(i)}(t)\right)^{a_i(t)} \tag{57}$$

where $\lambda_i$ are the eigenvalues of the integral operator $V(t)$, distributed with multiplicity $a_i(t)$. As far as $mt$ is finite, $V(t)$ is a square integrable operator possessing a discrete spectrum. However, in the short-distance limit, $mt \to 0$, this condition ceases to hold and the eigenvalues become dense in the interval $(-\infty, +\infty)$, with a multiplicity growing logarithmically as $\sim \ln\frac{1}{mt}$. Therefore, the disorder operator $\mu$ follows the leading power-law behaviour

$$\mu_0(t) \sim \frac{C}{(2t)^{x_\mu}} \tag{58}$$

with $x_\mu = -1/4$. This result is consistent with the intuitive idea that, upon approaching the impurity line in the ultra-violet limit, the operator $\mu$, characterized by the conformal weight $\left(-\frac{1}{8}, -\frac{1}{8}\right)$, starts interacting with its mirror image on the other side of the defect, along the identity channel. As a final remark, we hint at the possibility of sub-leading logarithmic corrections.

As regards the effects produced by the relevant perturbation, (52) behaves as

$$\begin{aligned} \mu_0(t;\kappa) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{\pi}\right)^n \int_{-\infty}^{+\infty} \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n \left[\prod_{k=1}^{n} \frac{\kappa\,\mathrm{e}^{-2mt\cosh\theta_k}}{2(\cosh\theta_k + \kappa)}\right] |A_n| \\ &= \det\left(1 + \frac{1}{\pi} V(t;\kappa)\right) \end{aligned} \tag{59}$$

with the kernel

$$V(\theta_i, \theta_j, t; \kappa) = \frac{e(\theta_i, t; \kappa) e(\theta_j, t; \kappa)}{2\cosh\frac{\theta_i+\theta_j}{2}} \qquad e(\theta, t; \kappa) = \sqrt{\frac{\kappa}{\cosh\theta + \kappa}}\, \mathrm{e}^{-mt\cosh\theta}. \tag{60}$$

In the short-distance limit, $|V|^2$ becomes unbounded, the leading singularity being dictated by the fixed boundary conditions' one. Thus we find the same critical exponent as in the previous case.

More interesting is the marginal situation. From general considerations extrapolated from the Ising model [4, 5], the non-universal nature of the marginal interaction is expected to affect the non-local sector of the theory, inducing a critical exponent continuously dependent on the coupling constant. Indeed, $\mu_0$ assumes the form

$$\begin{aligned}\mu_0(t;\chi) &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\sin^2\chi}{\pi}\right)^n \int_{-\infty}^{+\infty} \mathrm{d}\theta_1\ldots\mathrm{d}\theta_n \left[\prod_{k=1}^{n} \frac{\sinh^2\theta_k\ \mathrm{e}^{-2mt\cosh\theta_k}}{2(\cosh^2\theta_k - \sin^2\chi)}\right] |A_n| \\ &= \det\left(1 + \frac{\sin^2\chi}{\pi} V(t;\chi)\right) \end{aligned} \tag{61}$$

where

$$\begin{aligned} V(\theta_i,\theta_j,t;\chi) &= \frac{e(\theta_i,t;\chi)e(\theta_j,t;\chi)}{2\cosh\frac{\theta_i+\theta_j}{2}} \\ e(\theta,t;\chi) &= \sqrt{\frac{\cosh^2\theta - 1}{\cosh^2\theta - \sin^2\chi}}\ \mathrm{e}^{-mt\cosh\theta}. \end{aligned} \tag{62}$$

Repeating an analysis similar to that carried out for the fixed boundary condition, but this time with a parameter depending on the coupling constant, in front of the kernel in (61), we finally obtain the critical exponent

$$x_\mu = \frac{1}{4} - \frac{1}{2\pi^2}[\arccos^2(\sin^2\chi) + \arccos^2(-\sin^2\chi)]. \tag{63}$$

## 5. Final remarks

In this paper we have studied the effects induced by a defect interaction on the free theory of massive fermionic ghosts.

Working in the Lagrangian approach, we have dealt with two defect perturbations, respectively, of relevant and marginal nature. Explicit expressions for the reflection and transmission matrices have been derived. A careful analysis of their excitation spectra has pointed out the possibility of resonances and instabilities in the former case, and the occurrence of imaginary residues, relative to poles in the scattering amplitudes, in the latter one. Successively, we turned our attention to the exact computation of correlation functions, involving the most interesting operators in the theory, i.e. $\omega$, local in the ghost fields, and $\mu$, belonging to one of the non-trivial sectors of the model. In the marginal situation, a non-universal behaviour in the one-point function of the ‘ disorder’ operator $\mu$ has clearly emerged. Finally, the last appendix has been devoted to the analysis of the most general ‘disorder’ fields $\mu_\alpha$, characterized by non-locality index $\alpha$. The leading short-distance behaviour of their one-point function has been investigated by means of the ‘cluster’ expansion [33, 34].

It is worth noting that a delicate point of the present discussion concerns the comparison between the bootstrap approach and the Lagrangian description, in order to derive explicit expressions for the reflection and transmission amplitudes. In the former case, a richness of solutions descends but their physical explanation and ‘classification’, in terms of a fixed number of parameters related to the bulk S-matrix, results problematic. On the other hand, the Lagrangian approach, though subjected to the strong restriction of dealing only with local interactions, allows for a limited number of solutions, amenable to the easiest control. For instance, besides the defect perturbations already introduced, analysing other kinds of interactions could help in identifying new boundary conditions and, possibly, the operator content of the boundary theory.

Finally, we conclude with a remark on the simplified problem of a pure reflecting surface. As hinted at the end of the second section in relation to the free Dirac massive fermions, free theories, derived as the limit of interacting ones, admit a richer structure, as appears clearly in the bootstrap approach. It would be tempting, in this boundary case, to find an interacting theory, if any, behind the fermionic ghost model.

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## Appendix A

In this section, some useful results on the bulk system of fermionic ghosts are collected. The action is described by equation (9) where the symplectic form $J_{\alpha\beta}$ reads explicitly

$$J_{-+} = -J_{+-} = 1 \qquad J_{\alpha\gamma} J^{\gamma\beta} = \delta_\alpha^\beta \tag{64}$$

and the ghost fields $\Phi^\pm$, for later convenience, are redefined according to

$$\Phi^+ \to \Phi \qquad \Phi^- \to \bar{\Phi}.$$

The mode expansions for the components $\Phi_{(\pm)}$ and $\bar{\Phi}_{(\pm)}$, previously introduced (12), are

$$\begin{aligned} \Phi_{(\pm)}(x,t) &= \int \mathrm{d}\beta \big[\bar{a}_{(\pm)}(\beta)\ \mathrm{e}^{-\mathrm{i}m(t\cosh\beta - x\sinh\beta)} + a^\dagger_{(\pm)}(\beta)\ \mathrm{e}^{\mathrm{i}m(t\cosh\beta - x\sinh\beta)}\big] \\ \bar{\Phi}_{(\pm)}(x,t) &= \int \mathrm{d}\beta \big[-a_{(\pm)}(\beta)\ \mathrm{e}^{-\mathrm{i}m(t\cosh\beta - x\sinh\beta)} + \bar{a}^\dagger_{(\pm)}(\beta)\ \mathrm{e}^{\mathrm{i}m(t\cosh\beta - x\sinh\beta)}\big] \end{aligned} \tag{65}$$

where the creation and annihilation operators are subjected to the anti-commutation relations

$$\begin{aligned} &\big\{a_{(\pm)}(\beta), a^\dagger_{(\pm)}(\beta')\big\} = 2\pi\delta(\beta-\beta') \qquad \{a_{(\pm)}(\beta), a_{(\pm)}(\beta')\} = 0 = \big\{a^\dagger_{(\pm)}(\beta), a^\dagger_{(\pm)}(\beta')\big\} \\ &\big\{\bar{a}_{(\pm)}(\beta), \bar{a}^\dagger_{(\pm)}(\beta')\big\} = 2\pi\delta(\beta-\beta') \qquad \{\bar{a}_{(\pm)}(\beta), \bar{a}_{(\pm)}(\beta')\} = 0 = \big\{\bar{a}^\dagger_{(\pm)}(\beta), \bar{a}^\dagger_{(\pm)}(\beta')\big\}. \end{aligned} \tag{66}$$

Charge conjugation implemented on the Fock operators

$$\begin{aligned} \mathcal{C}a(\beta)\mathcal{C}^{-1} &= \bar{a}(\beta) \qquad & \mathcal{C}a^\dagger(\beta)\mathcal{C}^{-1} &= \bar{a}^\dagger(\beta) \\ \mathcal{C}\bar{a}(\beta)\mathcal{C}^{-1} &= -a(\beta) \qquad & \mathcal{C}\bar{a}^\dagger(\beta)\mathcal{C}^{-1} &= -a^\dagger(\beta) \end{aligned} \tag{67}$$

induces the following transformations on the ghost fields $\Phi \to \bar{\Phi}$ and $\bar{\Phi} \to -\Phi$. Finally, it is useful, for notational reasons, to identify the operator creating the bulk excitations with the excitations themselves

$$a^\dagger(\beta) \to A(\beta) \qquad \bar{a}^\dagger(\beta) \to \bar{A}(\beta). \tag{68}$$

## Appendix B

In this appendix we evaluate the critical exponent of the disorder operator $\mu$, corresponding to the fixed boundary conditions. Let us consider the logarithm of equation (57)

$$\ln \mu_0(t) = \sum_{i=1}^{\infty} a_i(t) \ln\left(1 + \frac{1}{\pi}\lambda^{(i)}(t)\right) \tag{69}$$

where, as explained before, $\lambda_i(t)$ are the eigenvalues of the integral operator $V(t)$, defined by equation (56). In the limit $mt \to 0$, such an operator turns out to be singular (it loses the property of square-integrability) and consequently, its eigenvalues become dense in $(-\infty, +\infty)$. The first problem to be faced concerns finding the exact solution to the eigenvalue equation

$$\int_{-\infty}^{+\infty} d\theta_2 \frac{1}{2\cosh\frac{\theta_1+\theta_2}{2}} \phi(\theta_2) = \lambda\phi(\theta_1) \tag{70}$$

which, after proper changes of variables, assumes definitely the form

$$\int_0^{\infty} du \frac{1}{uv+1} \xi(u) = \lambda\xi(v). \tag{71}$$

The peculiar expression of the new kernel $K(u,v) = \frac{1}{uv+1}$ suggests considering the Mellin transform of both sides of (71) [35, 36]. We finally end up with a simpler eigenvalue equation for the transformed quantities

$$(\lambda^2 - \tilde{K}(s)\tilde{K}(1-s))\tilde{\xi}(s) = 0 \tag{72}$$

where

$$\tilde{K}(s) = \frac{\pi}{\sin\pi s} \qquad 0 < \mathrm{Re}\, s < 1. \tag{73}$$

Some comments could be useful to evaluate the spectrum. Since the kernel is a symmetric function of its arguments and it is bounded, the spectrum has to be real and limited. Hence

$$\lambda_\pm(\tau) = \frac{\pm\pi}{\cosh\pi\tau} \qquad \tau \in (-\infty, +\infty). \tag{74}$$

Now equation (69) assumes the form

$$\ln\mu_0(t) = a(t)\int_{-\infty}^{\infty} d\tau \left[\ln\left(1 + \frac{1}{\pi}\lambda_+(\tau)\right) + \ln\left(1 + \frac{1}{\pi}\lambda_-(\tau)\right)\right] \tag{75}$$

where the multiplicity has been assumed to be uniform. Moreover, thanks to Mercer's theorem, $a(t) \sim \frac{1}{2\pi}\ln\frac{1}{t}$. At the end, the critical exponent is given by [37]

$$\begin{aligned} x_\mu &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau \left[\ln\left(1 + \frac{1}{\cosh\pi\tau}\right) + \ln\left(1 - \frac{1}{\cosh\pi\tau}\right)\right] \\ &= \frac{1}{\pi^2}\left[\frac{\pi^2}{4} - \frac{1}{2}(\arccos^2(1) + \arccos^2(-1))\right] = -\frac{1}{4}. \end{aligned} \tag{76}$$

## Appendix C

In this last appendix we discuss generic 'disorder' operators $\mu_\alpha$, which pick up a non-locality phase $e^{\pm 2\pi i\alpha}$, when they are taken around the ghost fields in the Euclidean plane

$$\begin{aligned} \Phi(z\, e^{2\pi i}, \bar{z}\, e^{-2\pi i})\mu_\alpha(0,0) &= e^{2\pi i\alpha}\Phi(z,\bar{z})\mu_\alpha(0,0) \\ \bar{\Phi}(z\, e^{2\pi i}, \bar{z}\, e^{-2\pi i})\mu_\alpha(0,0) &= e^{-2\pi i\alpha}\bar{\Phi}(z,\bar{z})\mu_\alpha(0,0). \end{aligned} \tag{77}$$

In particular, we are interested in deriving the leading short-distance behaviour of their one-point function in the case of fixed boundary conditions, in order to perform a comparison with the exact result previously obtained for the specific value $\alpha = \frac{1}{2}$.

The starting point is equation (52), where the form factors $f_n^{1/2}(-\beta_1, \ldots, \beta_n)$ must be replaced by the expression [25]

$$f_n^\alpha(-\beta_1, \ldots - \beta_n, \beta_1, \ldots, \beta_n) = (-)^{n(n+1)/2}(\sin\pi\alpha)^n\, e^{-(\alpha-\frac{1}{2})\sum_i^n 2\beta_i}|A_n| \tag{78}$$

with $|A_n|$ the determinant of the $n \times n$ matrix, whose components satisfy

$$A_{ij} = \frac{1}{\cosh \frac{\beta_i+\beta_j}{2}}. \tag{79}$$

In compact form, we can rewrite

$$\mu_0^\alpha(t) \equiv \langle \mu_\alpha(x,t)\rangle = \sum_{n=0}^{\infty} \frac{1}{n!}\int_{-\infty}^{+\infty} \mathrm{d}\beta_1 \ldots \mathrm{d}\beta_n \, \mathrm{e}^{-\rho \sum_j^n \cosh\beta_j} g_n^\alpha(\beta_1,\ldots,\beta_n) \tag{80}$$

where $\rho = 2mt$ and

$$g_n^\alpha(\beta_1,\ldots,\beta_n) \equiv \frac{1}{(2\pi)^n}(\sin\pi\alpha)^n \, \mathrm{e}^{-(\alpha-\frac{1}{2})\sum_j^n 2\beta_j}|A_n|. \tag{81}$$

These last two relations appear suitable to perform a ' cluster' expansion, according to the technique exposed, for instance, in [33, 34]. Therefore, the logarithm of (80) assumes the form

$$\ln \mu_0^\alpha(t) = \sum_{n=1}^{\infty} \frac{1}{n!}\int_{-\infty}^{+\infty} \mathrm{d}\beta_1 \ldots \mathrm{d}\beta_n \, \mathrm{e}^{-\rho \sum_j^n \cosh\beta_j} h_n^\alpha(\beta_1,\ldots,\beta_n) \tag{82}$$

where the functions $h_n^\alpha$ are proper combinations of the $g_n^\alpha$. For our purposes, we need only the first few relations, which read explicitly [34]

$$\begin{aligned} g_1^\alpha &= h_1^\alpha \\ g_{12}^\alpha &= h_{12}^\alpha + h_1^\alpha h_2^\alpha \\ g_{123}^\alpha &= h_{123}^\alpha + h_{12}^\alpha h_3^\alpha + h_{23}^\alpha h_1^\alpha + h_{31}^\alpha h_2^\alpha + h_1^\alpha h_2^\alpha h_3^\alpha. \end{aligned} \tag{83}$$

The key point of the standard 'cluster' expansion is that, since the functions $h_n$ depend only on rapidity differences, they contain a redundant variable. Thus, it is possible, at all orders, to extract the integral

$$\int_0^{+\infty} \mathrm{d}\beta \, \mathrm{e}^{-\rho\cosh\beta} = K_0(\rho) \tag{84}$$

which is responsible for the logarithmic behaviour $\ln\frac{1}{\rho}$, as $\rho \to 0$. The remaining integrals multiplying such a result,

$$2\sum_{n=1}^{\infty}\frac{1}{n!}\int_{-\infty}^{+\infty} \mathrm{d}\beta_1 \ldots \mathrm{d}\beta_{n-1} h_n^\alpha(\beta_1,\ldots,\beta_{n-1},0) \tag{85}$$

give the approximate value of the critical exponent, provided that the 'cluster' condition

$$h_n(\beta_1,\ldots,\beta_n) = \mathcal{O}(\mathrm{e}^{-|\beta_i|}) \tag{86}$$

is fulfilled, for Re $\beta_i \to \pm\infty$.

On the other hand, the fermionic ghost model displays a substantial difference. The functions $h_n^\alpha$ depend, by construction, on the sum of rapidities. Thus, only contributions of even order in the series (82) admit a redundant variable, finally leading to a logarithmic behaviour. The remaining terms, of odd order, provide convergent pieces, useful to reconstruct the normalization constant of the one-point function.

In order to study explicitly the short-distance behaviour of $\mu_0^\alpha$, we focus our attention on the second-order contribution. All we need to know is

$$h_{12}^\alpha(\beta_1,\beta_2) = -\left(\frac{\sin\pi\alpha}{2\pi}\right)^2 \frac{\mathrm{e}^{-(2\alpha-1)(\beta_1+\beta_2)}}{\left(\cosh\frac{\beta_1+\beta_2}{2}\right)^2}. \tag{87}$$

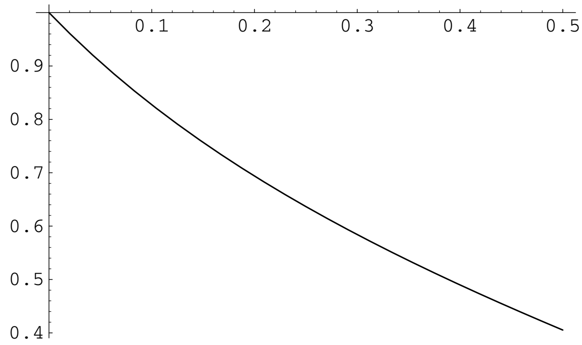


**Figure 1.** $-\frac{x_\alpha}{\alpha/2}$ as a function of the non-locality index $\alpha$, for $\alpha \in [0, \frac{1}{2}]$.

Hence, substituting in (82), after straightforward calculations, we finally end up with

$$\ln \mu_0^\alpha(t) = x_\alpha \ln \frac{1}{2mt} \tag{88}$$

where the critical exponent reads

$$x_\alpha = -\frac{1-2\alpha}{2\pi} \tan(\pi\alpha). \tag{89}$$

For small values of the non-locality index, $x_\alpha \to -\alpha/2$. However, we are mainly interested in the limit $\alpha \to 1/2$, where a comparison with the exact value $x_{1/2} \equiv x_\mu = -1/4 = -0.25$, previously derived, is possible. Equation (89) leads to the result $x_{1/2} \sim -1/\pi^2 \sim -0.10$, independent of $\alpha$. This large discrepancy suggests that the 'cluster' approximation, for this particular non-locality index, fails to reproduce the exact critical exponent with accuracy, but, nevertheless, hints at its correct sign. Finally, figure 1 displays the ratio $\frac{-x_\alpha}{\alpha/2}$, in order to make visible deviations from the small-$\alpha$ behaviour.

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